

UNIVERSAL NONLINEAR SMALL-SCALE DYNAMO

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ABSTRACT

We consider astrophysically relevant nonlinear MHD dynamo at large Reynolds numbers (Re). We argue that it is universal in a sense that magnetic energy grows at a rate which is a constant fraction C_E of the total turbulent dissipation rate. On the basis of locality bounds we claim that this “efficiency of small-scale dynamo”, C_E , is a true constant for large Re and is determined only by strongly nonlinear dynamics at the equipartition scale. We measured C_E in numerical simulations and observed a value around 0.05 in highest resolution simulations. We address the issue of C_E being small, unlike Kolmogorov constant which is of order unity.

1. INTRODUCTION

MHD turbulence is ubiquitous in astrophysical and space environments (Goldstein et al 1995; Armstrong et al 1995). Reynolds numbers are, typically, very high, owing to large scales of astrophysical processes compared to small dissipative scales. One of the central questions of MHD dynamics is how initially unmagnetized well-conductive fluid generates its own magnetic field, known broadly as “dynamo”. Turbulent dynamo has been roughly subdivided into “large-scale dynamo” and “small-scale” or “fluctuation dynamo” depending on whether magnetic fields are amplified on scales larger or smaller than turbulence outer scales. Although a few “no-dynamo” theorems has been proved for flows with symmetries, a generic turbulent flow, which possess no exact symmetries was expected to amplify magnetic fields by stretching, due to particle separation in a turbulent flow. For the large-scale field, a “twist-stretch-fold” mechanism was introduced (Vainshtein & Zeldovich 1972). Turbulent flow possessing perfect statistical isotropy can not generate large-scale magnetic field. Observed large-scale fields, such as in disk galaxies, are generated when statistical symmetries of turbulence are broken by large-scale asymmetries of the system, such as stratification, rotation and shear (see, e.g., Vishniac & Cho 2001; Käpylä et al 2009). Since these symmetries are only weakly broken, large-scale dynamo is slow. Small-scale dynamo does not suffer from this restriction and can be fast. Kinematic small-scale dynamo, which ignores the backreaction of the magnetic field has been studied extensively before (Kazantsev 1968; Kulsrud & Anderson 1992). However, kinematic dynamo is irrelevant for astrophysical environments, because it grows exponentially and becomes inapplicable at smallest turbulent timescales which are tiny by astrophysical standards. Additionally, kinematic dynamo have positive spectral index (typically 3/2) at all scales, which is incompatible with observations in galaxy clusters (Laing et al 2008) which clearly indicate steep spectrum with negative power index at small scales. Due to preexisting astrophysical fields small-scale dynamo starts in nonlinear regime. It was discovered numerically that small-scale dynamo continues to grow after kinematic stage, producing steep spectrum at small scales and signifi-

cant outer-scale fields (Haugen et al 2004; Brandenburg & Subramanian 2005; Cho et al 2009; Ryu et al 2008). Furthermore, MHD turbulence produces turbulent diffusivity (aka “ β -effect”), which is essential for large-scale dynamo (Käpylä et al 2009) and reconnection (Lazarian & Vishniac 1999; Eyink 2010). Saturation of small-scale dynamo seems to be independent on Re and Pr as long as Re is large (Haugen et al 2004) and the magnetic energy growth rate could be constant (Schekochihin & Cowley 2007; Cho et al 2009; Beresnyak et al 2009; Ryu et al 2008). Small-scale dynamo is faster than large-scale dynamo in most astrophysical environments and magnetic energy grows quickly to equipartition with kinetic motions, with the largest scales of such field being a fraction of the outer scale of turbulence. Subsequently, these turbulent fields are slowly ordered by mean-field dynamo, with turbulent diffusivity of MHD turbulence playing essential role. In this paper we provide sufficient analytical and numerical argumentation behind the universality of nonlinear small-scale dynamo.

2. NONLINEAR SMALL-SCALE DYNAMO

We assume that the spectra of magnetic and kinetic energy at a particular moment of time are similar to what is presented on Fig. 1. Magnetic and kinetic spectra cross at some “equipartition” scale $1/k^*$, below which both spectra are steep due to MHD cascade (see, e.g., Goldreich & Sridhar 1995; Beresnyak 2011). This is suggested by both numerical evidence (Beresnyak & Lazarian 2009b; Cho et al 2009) observations of magnetic fields in clusters of galaxies (Laing et al 2008). At larger scales magnetic spectrum is shallow, k^α , $\alpha > 0$, while kinetic spectrum is steep due to a hydro cascade. Most of the magnetic energy is concentrated at scale $1/k^*$. We designate C_K and C_M as Kolmogorov constants of hydro and MHD respectively. The hydrodynamic cascade rate is ϵ and the MHD cascade rate as ϵ_2 . Due to the conservation of energy in the inertial range, magnetic energy will grow at a rate $\epsilon - \epsilon_2$. We will designate $C_E = (\epsilon - \epsilon_2)/\epsilon$ as an “efficiency of small-scale dynamo” and we will argue that this is a true constant, since: a) turbulent dynamics is local in scale in the inertial range; b) neither ideal MHD nor Euler equations contain any scale explicitly. Magnetic energy, therefore, grows linearly with time if

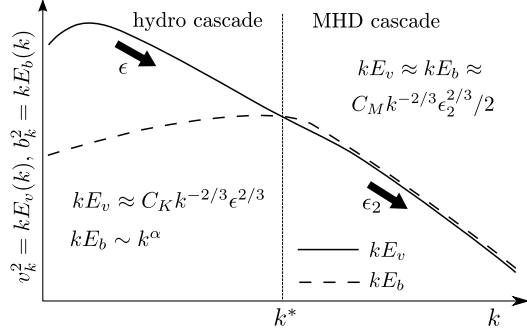


FIG. 1.— A cartoon of kinetic and magnetic spectra in small-scale dynamo, at a particular moment of time when equipartition wavenumber is k^* .

$\epsilon = \text{const.}$ The equipartition scale $1/k^*$ will grow with time as $t^{3/2}$ (Beresnyak et al 2009). This is equivalent to saying that small-scale dynamo saturates at several dynamical times at scale $1/k^*$ and proceeds to a twice larger scale (Schekochihin & Cowley 2007). If magnetic energy grows approximately till equipartition (Haugen et al 2004; Cho et al 2009), the whole process will take around several dynamical timescales of the system, or more quantitatively, $(C_K^{3/2}/C_E)(L/v_L)$.

3. LOCALITY OF SMALL-SCALE DYNAMO

We will use “smooth filtering” approach with dyadic-wide filter in k -space (Aluie & Eyink 2010; Eyink 2005). We designate a filtered vector quantity as $\mathbf{a}^{[k]}$ where k is a center of a dyadic Fourier filter in the range of wave numbers $[k/2, 2k]$. The actual logarithmic width of this filter is irrelevant to further argumentation, as long as it is not very small. We will assume that the vector field \mathbf{a} is Hölder-continuous with some exponent and designate $a_k = \langle |\mathbf{a}^{[k]}|^3 \rangle^{1/3}$ which has to scale as k^{σ_3} , e.g., $k^{-1/3}$ for velocity in Kolmogorov turbulence. The energy cascade rate is $\epsilon = C_K^{-3/2} k v_k^3$, where here we defined Kolmogorov constant C_K by third order, rather than second order quantities. We will keep this designation, assuming that traditional Kolmogorov constant could be used instead. We use spectral shell energy transfer functions such as $T_{vv}(p, k) = -\langle \mathbf{v}^{[k]} (\mathbf{v} \cdot \nabla) \mathbf{v}^{[p]} \rangle$, $T_{w^+w^+}(p, k) = -\langle \mathbf{w}^{+ [k]} (\mathbf{w}^+ \cdot \nabla) \mathbf{w}^{+ [p]} \rangle$ (Alexakis et al 2005), applicable to incompressible ideal MHD equations, where w^\pm are Elsässer variables and v, b and w^\pm are measured in the same Alfvénic units. We will use central frequency k and study “infrared” (IR) transfers from $p \ll k$, and “ultraviolet” (UV) transfers, from $q \gg k$. We will provide bounds on $|T|$, in units of energy rate as in Aluie & Eyink (2010); Eyink (2005) and relative volume-averaged bounds which are divided the actual energy rate and are dimensionless. We will consider three main wavevector intervals presented on Fig. 1: $k \ll k^*$ (“hydro cascade”), $k \sim k^*$ (“dynamo”) and $k \gg k^*$ (“MHD cascade”).

4. MHD CASCADE, $K \gg K^*$.

The only energy cascades here are Elsässer cascades and, by design of our problem, w^+ and w^- have the same statistics, so we will drop \pm . For an exchange with $p \ll k$ band, for $|T_{ww}|$, using Hölder inequality and wavenumber conservation we get an upper bound of $p v_p w_k^2$ and for $q \gg k$ band it is $k w_q^2 w_k$, these bounds are asymptoti-

TABLE 1
TRANSFERS AND UPPER LIMITS

Transfers	$p \ll k$	$q \gg k$
$T_{vv}(p, k) = -\langle \mathbf{v}^{[k]} (\mathbf{v} \cdot \nabla) \mathbf{v}^{[p]} \rangle$	$p v_p v_k^2$	$k v_k v_q^2$
$T_{bb}(p, k) = -\langle \mathbf{b}^{[k]} (\mathbf{b} \cdot \nabla) \mathbf{b}^{[p]} \rangle$	$p v_p v_k b_k$	$k b_k v_q b_q$
$T_{vb}(p, k) = \langle \mathbf{b}^{[k]} (\mathbf{b} \cdot \nabla) \mathbf{v}^{[p]} \rangle$	$p v_p b_k^2$	$k b_k v_q b_q$
$T_{bv}(p, k) = \langle \mathbf{v}^{[k]} (\mathbf{b} \cdot \nabla) \mathbf{b}^{[p]} \rangle$	$p b_p v_k b_k$	$k v_k b_q^2$
$T_{w^+w^+}(p, k) = -\langle \mathbf{w}^{+ [k]} (\mathbf{w}^+ \cdot \nabla) \mathbf{w}^{+ [p]} \rangle$	$p w_p w_k^2$	$k w_k w_q^2$

cally small, see Eyink (2005). For the full list of transfers and limits refer to Table 1. The relative bound should be taken with respect to $C_M^{-3/2} k w_k^3$, where C_M is a Kolmogorov constant for MHD, from which we get that most of the energy transfer with the $[k]$ band should come from $[k C_M^{-9/4}, k C_M^{9/4}]$ band, see Beresnyak (2011). The global transfers between kinetic and magnetic energy must average out in this regime, nevertheless, the pointwise IR and UV transfers can be bounded by $p b_p v_k b_k$ and $k v_k b_q^2$ and are small (Eyink 2005).

5. HYDRO CASCADE, $K \ll K^*$

Despite having some magnetic energy at these scales, most of the energy transfer is dominated by velocity field. Indeed, $|T_{vv}|$ is bounded by $p v_p v_k^2$ for $p \ll k$ and by $k v_q^2 v_k$ for $q \gg k$. Compared to these, $|T_{bv}|$ transfers are negligible: $p b_p v_k b_k$ and $k b_q^2 v_k$. For magnetic energy in $p \ll k$ case we have $|T_{vb}|$ and $|T_{bb}|$ transfers bounded by $p v_p b_k^2$, $p b_p v_k b_k$ and for $q \gg k$ case $|T_{vb}|$ and $|T_{bb}|$ are bounded by $k b_k v_q b_q$. Out of these three expressions the first two go to zero, while the third goes to zero if $\alpha - 2/3 < 0$ or have a maximum at $q = k^*$ if $\alpha - 2/3 > 0$. This means that for the transfer to magnetic energy we have IR locality, but not necessarily UV locality. Note that magnetic energy for $k \ll k^*$ is small compared to the total, which is dominated by $k = k^*$. We will assume that $\alpha - 2/3 > 0$ and that the spectrum of b_k for $k < k^*$ is formed by nonlocal $|T_{vb}|$ and $|T_{bb}|$ transfers from k^* , namely magnetic structures at k are formed by stretching of magnetic field at k^* by velocity field at k . Magnetic spectrum before k^* is, therefore, nonlocal and might not be a power-law.

6. DYNAMO CASCADE $K = K^*$

In this transitional regime our estimates of Elsässer UV transfer and kinetic IR transfer from two previous sections will hold. We are interested how these two are coupled together and produce observable magnetic energy growth or decay. IR $p \ll k^*$ $|T_{vb}|$ and $|T_{bb}|$ transfers will be bounded by $p v_p b_k^2$ and $p b_p v_k b_k$, which go to zero, so there is a good IR locality. Ultraviolet transfers will be bounded by $k^* b_k b_q v_q$. This quantity also goes to zero as q increases, so there is an UV locality for this regime as well. Let us come up with bounds of relative locality. Indeed, the actual growth of magnetic energy was defined as $\epsilon_B = \epsilon - \epsilon_2 = C_E C_K^{-3/2} k v_k^3$. So, $p \ll k^*$ IR bound is $k^* C_E^{3/2} C_K^{-9/4}$ and UV bound is $k^* C_E^{-3/2} C_M^{9/4}$. We conclude that most of the interaction which result in magnetic energy growth must reside in the wavevector interval of $k^* [C_E^{3/2} C_K^{-9/4}, C_E^{-3/2} C_M^{9/4}]$. Numerically, if we substitute $C_K = 1.6$, $C_M = 4.2$, $C_E = 0.05$ we get

TABLE 2
THREE-DIMENSIONAL MHD SIMULATIONS

Run	n	N^3	Dissipation	$\langle \epsilon \rangle$	Re	C_E
M1-6	6	256^3	$-7.6 \cdot 10^{-4} k^2$	0.091	1000	0.031 ± 0.002
M7-9	3	512^3	$-3.0 \cdot 10^{-4} k^2$	0.091	2600	0.034 ± 0.004
M10-12	3	1024^3	$-1.2 \cdot 10^{-4} k^2$	0.091	6600	0.041 ± 0.005
M13	1	1024^3	$-1.6 \cdot 10^{-9} k^4$	0.182	—	0.05 ± 0.005
M14	1	1536^3	$-1.5 \cdot 10^{-15} k^6$	0.24	—	0.05 ± 0.005

the interval of $k^*[0.004, 2000]$. So, despite being asymptotically local, small-scale dynamo can be fairly nonlocal from practical standpoint.

To summarize, kinetic cascade at large scales and MHD cascade at small scales in the inertial range are dominated by local interactions. The transition between kinetic cascade and MHD cascade is also dominated by local interactions, and since ideal MHD equations do not contain any scale explicitly, the efficiency of small-scale dynamo C_E is a true universal constant. Note that C_E relates energy fluxes, not energies, so this claim is unaffected by the presence of intermittency. Magnetic spectrum at $k \ll k^*$ is dominated by nonlocal triads that reprocess magnetic energy from $k = k^*$ and, since this part of the spectrum contains negligible magnetic energy, our universality claim is unaffected by this nonlocality.

7. NUMERICAL RESULTS

We performed numerical simulations of statistically homogeneous isotropic small-scale dynamo by solving MHD equations with stochastic non-helical driving and explicit dissipation with $Pr_m = 1$. The details of the code and driving are described in detail in our earlier publications (Beresnyak 2011; Beresnyak & Lazarian 2009) and Table 1 shows simulation parameters. We started each simulation from previously well-evolved driven hydro simulation by seeding low level white noise magnetic field. We ran several statistically independent simulations in each group and obtained growth rates and errors from statistical (or sample) averages. In all simulations, except M14, the energy injection rate was controlled. Fig 2 shows sample-averaged time evolution of magnetic energy. Growth is initially exponential and smoothly transition into the linear stage. Note, that scatter is initially small, but grows with time, which is consistent with the picture of magnetic field growing at progressively larger scales and having progressively less independent realizations in a single datacube.

8. EFFICIENCY OF SMALL-SCALE DYNAMO

Our C_E is much smaller than unity. One would expect a quantity of order unity because this is a universal number, determined only by strong interaction on equipartition scale. If we refer to the ideal incompressible MHD equations, written in terms of Elsässer variables, $\partial_t \mathbf{w}^\pm + \hat{S}(\mathbf{w}^\mp \cdot \nabla) \mathbf{w}^\pm = 0$, dynamo could be understood as decorrelation of \mathbf{w}^\pm which are originally perfectly correlated with $\mathbf{w}^+ = \mathbf{w}^-$ in the hydrodynamic cascade. In our case this decorrelation is happening at the equipartition scale k^* . Being time-dependent, this decorrelation propagates upscale, while ordinary energy cascade goes downscale. The small value of C_E might be due to this. A different viewpoint on small-scale dynamo is a picture when magnetic field is being stretched by kinetic motions

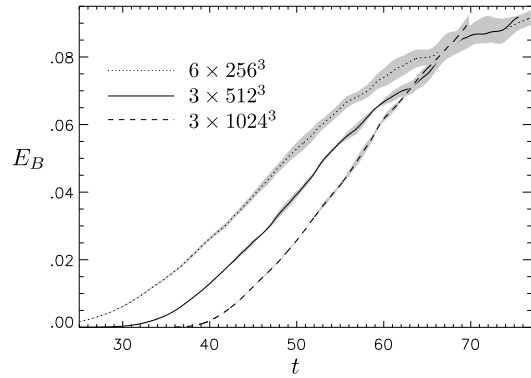


FIG. 2.— Magnetic energy growth observed in simulations M1-6, M7-9 and M10-13. We used sample averages which greatly reduced fluctuations and allowed us to measure C_E with sufficient precision.

and, therefore, amplified. With small or negligible diffusion the field becomes extremely tangled (Schekochihin et al 2004), quite contrary to the observations and numerical experiments, where magnetic field has an energy-containing scale which is larger than the scale of resistive dissipation. In the nonlinear regime, therefore, the intuitive picture with stretching with resistive diffusion is completely wrong. It is inevitable to appeal to turbulent diffusion which helps to create larger-scale field. Both stretching and diffusion depend on turbulence properties at the same designated scale $1/k^*$, therefore in the asymptotic regime of large Re one of these processes must dominate. As we know C_E is small so stretching and diffusion are close to canceling each other.

9. KINEMATIC DYNAMO RATES

A better studied and understood kinematic dynamo might shed some light on the problem of small C_E . In the kinematic regime, when we neglect Lorentz force in the MHD equation, the growth is exponential and the rate is expected to come from fastest shearing rates of smallest turbulent eddies. Observed rates, however, are smaller which was interpreted as competition between stretching and turbulent mixing (Eyink et al 2011). In our simulations, in the kinematic regime of M7-9 we observed growth rate $\gamma\tau_\eta = 0.0326$, where $\tau_\eta = (\nu/\epsilon)^{1/2}$ is a Kolmogorov timescale, which is consistent with (Haugen et al 2004; Schekochihin et al 2004). With minimum timescale, $\tau_{\min} \approx 9\tau_\eta$, $\gamma\tau_{\min} = 0.3$, which is still small.

Kazantsev-Kraichnan model (Kazantsev 1968; Kulsrud & Anderson 1992) predicts $\gamma\tau_{\min} \sim 1$. This model, however, uses ad-hoc delta-correlated velocity which does not correspond to any dynamic turbulence and its statistics is time-reversible as opposed to time-irreversible real turbulence. Time irreversibility of hydro turbulence actually mandates that fluid particles separate faster backwards in time, since $\langle v_{\parallel}^3 \rangle = -4/5\epsilon l$ is negative.

In order to study the interplay of stretching and diffusion, we performed several simulations of kinematic dynamo forward and backward in time. We followed full three-dimensional evolution of \mathbf{v} and \mathbf{b} and approximated “backward in time” by reversing velocity direction. Initial condition for magnetic field was typically random noise. Since we couldn’t reverse viscous losses in DNS, we used viscosity $\nu = 0$, but magnetic diffusivity $\eta > 0$. In the first set of simulations we set initial velocity as \mathbf{v} and $-\mathbf{v}$ from evolved viscous runs. The growth rates is

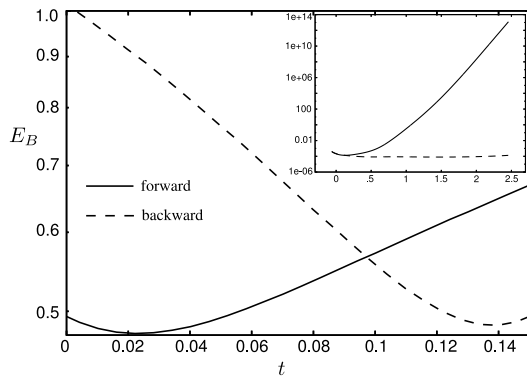


FIG. 3.— Evolution of magnetic energy forward and backward in time (dashed is with reversed x-axis). Inset: a naive simulation with initial state $+\mathbf{v}$ and $-\mathbf{v}$ from Navier-Stokes simulation (not a rigorous backward in time simulation).

shown on inset of Fig. 3. Quite surprisingly, the “backward” simulation did not produce any growth for several dynamical times. Unexpectedly, simply reversing velocity has such a profound impact on kinematic dynamo, despite spectra being very close to each other, suggesting that it is not only the spectrum that determines growth but rather the actual statistical properties of velocity, which will determine whether stretching or diffusion wins, i.e. if there is a dynamo or a no-dynamo, even in the simple kinematic case. In this simulation we observed a typical k^2 “thermal pool” at the end of velocity spectrum which had shortest timescales. “Thermal pool” was clearly time-irreversible, unlike the true physical thermal pool, consistent with (Ray et al 2011).

The next series of simulations were reproducing an actual backward in time dynamics. In order to achieve this we evolved initial state for a fairly short time with $\nu = 0$ and then we evolved it for the same time reverting velocity with $\nu = 0$ and confirmed that final state is close to initial state, due to reversibility of truncated Euler equations. The results for dynamo growth is shown of Fig. 3. We see that backward dynamo is faster by a factor of 2.0 ± 0.1 , which is actually consistent with the ratio of particle diffusion forward and backward in time in (Eyink 2010). This result again reinforces our statement that dynamo is a result of competing mechanisms

of turbulent stretching and turbulent diffusion and the outcome depend on statistics of velocity other than just velocity spectrum.

10. DISCUSSION

Linear growth regime that we considered in this Letter provides a fastest growth rate for magnetic energy, $C_E \epsilon$, which is actually just a constant fraction of absolute energetic limit of ϵ . This linear growth starts with $t = 0$ in astrophysical environments with very high Re, as long as turbulence is well-developed. Exponential growth regime, that is relevant for finite Re and small initial seed fields provide much slower growth. Combined with finite time that is available in simulated astrophysical objects this may result in severe underestimation of the magnetic energy and, therefore, of dynamical importance of magnetic fields. For example, cosmological simulations of halo collapse, that have rather modest Re, could be underestimating magnetic energy (Sur et al. 2010). A resolution study must confirm convergence of magnetic energy with resolution. If exponential growth is observed, magnetic energy is underestimated.

In this Letter we discussed the issue of C_E being small and interpreted this as a competition between turbulent stretching that enhances magnetic field and turbulent diffusion that tries to dissipate it. Indeed, the negative values of C_E , from theoretical viewpoint, are also allowed. This would be a situation of non-dynamo in the nonlinear regime. Isotropic homogeneous Kolmogorov turbulence does produce a growth and most astrophysical environments that are well-magnetized. However, turbulent magnetic diffusion can dominate in situations when turbulence is very different from isotropic and homogeneous and have finite Re. This might explain why dynamo experiments sometimes generate larger fields in the absence of turbulence (Colgate et al. 2011).

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